MATH 1450 EXAM2

NAME Key - GRADE ———-OUT OF 15 PTS

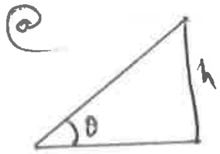
Answer each of following questions correctly for a full credit.

1. **(4.5pts) Related Rates:** Answer each question $(a - c)$ separately (up to three decimal places.)

   (a) A hot air balloon rising vertically is tracked by an observer located 4km from the lift-off point. At a certain moment, the angle between the observer's line of sight and the horizontal is $\frac{\pi}{5}$ and it is changing at a rate of 0.4 rad/hr. How fast is the balloon rising at this moment?

   (b) Water pours into a conical tank of height 10 m and radius 4 m at a rate of 6 m^3/min. (i) At what rate is the water level rising when the level is 5 m high? (ii) As time passes, what happens to the rate at which the water level rises?

(a)

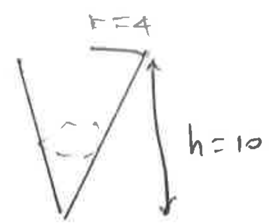


$\tan\theta = \frac{y}{4} \rightarrow \sec^2\theta \frac{d\theta}{dt} = \frac{1}{4}\frac{dy}{dt}$

$\rightarrow \frac{dy}{dt} = 4\left(\frac{d\theta}{dt}\right)\sec^2\theta$

$= 4(0.4)\frac{1}{\cos^2(\frac{\pi}{5})} \approx \boxed{2.445 \text{ km/hr}}$

(b)



$\frac{r}{h} = \frac{10}{4} = \frac{5}{2} \rightarrow r = \frac{2}{5}h$

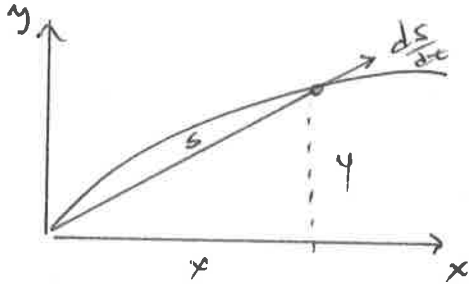
$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi\left(\frac{2}{5}h\right)^2 h = \frac{4\pi}{75}h^3$

So $\frac{dV}{dt} = \frac{4\pi h^2}{25}\frac{dh}{dt} \rightarrow \frac{dh}{dt} = \frac{25}{4\pi h^2}\frac{dV}{dt}$. When $h = 5$, $\frac{dV}{dt} = 6$

Hence $\frac{dh}{dt} = \frac{25}{\pi(5)^2}(6) = \frac{3}{2\pi} \approx \boxed{.477 \text{ rad/min}}$

(ii) because $\frac{dh}{dt} = \frac{k}{h^2}$, ie is inversely proportional to h^2, as h increases, $\frac{dh}{dt}$ decreases, (until it hits zero.)

(c) A particle is moving along the curve $y = \sqrt{x}$. As the particle passes through the point $(4, 2)$, its x–coordinate is increasing at a rate of 3 cm/sec. How fast is the distance from the particle to the origin changing at this instant? (up to *three decimals*).



$$y = \sqrt{x} \rightarrow \frac{dy}{dt} = \frac{1}{2\sqrt{x}} \frac{dx}{dt}$$

$$\rightarrow \frac{dy}{dt} = \frac{1}{2\sqrt{4}}(3) = \frac{3}{4} \text{ cm/s}$$

When $x = 4$, $y = 2 \rightarrow s = \sqrt{16+4} = 2\sqrt{5}$

Further, $s^2 = x^2 + y^2 \rightarrow \frac{ds}{dt} = \frac{1}{s}\left[x\frac{dx}{dt} + y\frac{dy}{dt}\right]$

$$= \frac{1}{2\sqrt{5}}\left[12 + \frac{3}{2}\right] = \frac{27\sqrt{5}}{20} \text{ cm/s}$$

$$\boxed{\simeq 3.02 \text{ cm/s}}$$

2. **(4.5pts) Approximation and Newton's method:**

   (a) Compute the linearization of $f(x) = e^{x-1}$ at $c = 1$.

   (b) Calculate (up to *five decimals*) the first three approximations x_1, x_2, x_3 to a root of $g(x) = x^2 - 5$ using the initial guess $x_0 = 2$.

(a) $f(x) = e^{x-1}$ $f'(x) = e^{x-1} \rightarrow f'(1) = e^0 = 1$ and

$f(1) = e^0 = 1$

So $L(x) \doteq f'(1)(x-1) + f(1)$

$= 1(x-1) + 1$ & $\boxed{f(x) \simeq L(x) = x}$

(b) $x_{n+1} = x_n - \frac{g(x_n)}{g'(x_n)} = x_{n+1} - \frac{x_n^2 - 5}{2x_n}$ for $n = 0, 1, 2$

$x_0 = 2 \rightarrow x_1 = 2 - \frac{(2^2-5)}{2(2)} = \boxed{2.25}$

$$\boxed{x_2 = 2.23611}$$

$$\boxed{x_3 = 2.23606}$$

(c) Find the linear approximation to $f(x) = \ln(x)$ near 1.

$$L(x) \simeq f(1) + f'(1)(x-1)$$

$$\simeq 0 + 1(x-1) = x-1$$

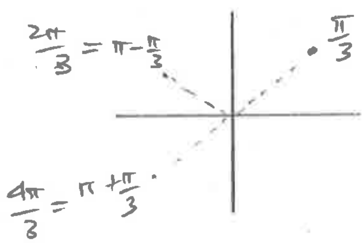
3. (6pts) **First and Second Derivatives Tests**

(a) Find the open intervals where $g(x) = x + 2sin(x)$ is monotonic, i.e., increasing or decreasing, with $0 < x < 2\pi$.

(b) Find all critical and the extreme values of the function $f(x) = 2x^3 - 9x^2 + 12x$ on the interval $[0, 3]$.

ⓐ $g(x) = x + 2\sin x \longrightarrow g'(x) = 1 + 2\cos x = 0$

$\longrightarrow \cos x = -1/2 \longrightarrow x = 2\pi/3 \,;\, \frac{4\pi}{3}$



x	0		$\frac{2\pi}{3}$		$\frac{4\pi}{3}$		2π
Sign of g'		$+$		$-$		$+$	
behavior of g		↗		↘		↗	

g is increasing over $\left(0, \frac{2\pi}{3}\right) \cup \left(\frac{4\pi}{3}, 2\pi\right)$

g is decreasing over $\left(\frac{2\pi}{3}, \frac{4\pi}{3}\right)$

ⓑ $f'(x) = 6x^2 - 18x + 12 = 0 \longrightarrow x = 1, 2$ (critical values)

$f(1) = 5 \,;\, f(2) = 4 \,;\, f(0) = 0 \,;\, f(3) = 9$

The min value : 0

The max value : 9

(c) Answer the following questions for the given function $f(x) = 3x - x^3$.

i- Find the critical *points.*

ii- Find where (*in an interval form*) the function is increasing / decreasing.

iii- Find the relative maxima and minima, if any.

iv- Find where (*the value(s) of x*) the second derivative is zero or undefined.

v- Find where (*in an interval form*) the function is concave up / down.

vi- Find the inflection *point(s)*, if any.

(i) $f'(x) = 3 - 3x^2 = 0 \longrightarrow x = \pm 1$ (critical values)

The critical points are $\boxed{(1,2) \text{ and } (-1,-2)}$

(ii)

x:	$-\infty$	-1		1	$+\infty$
Sign of $f'(x)$	$-$		$+$		$-$
	$\searrow$		$\nearrow$		$\searrow$

Increasing $(-1, +1)$

Decreasing $(-\infty, -1) \cup (1, +\infty)$

(iii) Relative min $\boxed{(-1, -2)}$

Relative max $\boxed{(1, 2)}$

(iv) $f''(x) = 0 \longrightarrow -6x = 0 \longrightarrow \boxed{x = 0}$

(v)

	$-\infty$	0	$+\infty$
Sign of f'':	$+$		$-$
	$\smile$		$\frown$

Concave up $\boxed{(-\infty, 0)}$

Concave down $\boxed{(0, +\infty)}$

(iv) Inflection point: $\boxed{(0,0)}$